INTRODUCTION

TO

LUCICAL ENGINEERING

BY

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NOTE.

The following has been written for the use of the Inspection Department of the American Luxfer Prism Company. Each Inspector is expected to become familiar with this treatment before entering upon his work, and to use it for reference in his inspection work.

INDEX OF REFRACTION.

The ordinary velocity of light is something like 186,000 miles a second. This is the velocity in vacuo, *e. g.*, as light travels from the sun to the earth. If the light passes through any medium other than that which is known as the ether, the velocity of the light is different from the value given above, and sometimes is very different. In all known media, the velocity of light is less than in vacuo. The ratio of the velocity of light in vacuo to the velocity of light in any particular substance is called the index of refraction of that substance. This index of refraction varies for different colors of light, the blue light traveling slower in dense media than the red, so that the index of refraction for blue light is greater than that for yellow or red light. It is customary to give the index of refraction for the yellow rays as the index of refraction of any body. The index of refraction for the glass of which Luxfer Prisms are made, is about 1.53, *i. e.*, the velocity of yellow light in this material is about two-thirds that in vacuo or about 120,000 miles a second. The index of refraction for air is about 1.0003, which may be regarded as unity for all our purposes.

It is a matter of every day experience that the direction of a ray of light through ordinary media is a straight line. It is in recognition of this fact that the savage has learned to aim his arrow and the modern marksman his long distance rifle. This truth was realized long before the nature of light was suspected. Modern science has shown light to consist of a series of waves which are perpendicular to the direction of propagation, or to the ray itself; and, if we could see the waves, their cross-section would probably remind us very much of the waves coming in on the lake shore. What is emphasized here is that the wave-fronts are perpendicular to the direction in which they travel.

When light passes from one medium to another we have seen above that its velocity is changed. Its direction is also changed. We shall now see that this change of direction of the ray is a result of the change of velocity.

In Fig. 1 let J-K and A-M represent waves of light moving in the direction indicated by the arrows, and falling upon the surface of glass S-T. In a short time after the wave has taken the position A-M, the part of the wave which was at C, will have passed through the distance C-B with the velocity of light in air. During the same time the part of the wave which was at A will have passed through a distance about $2/3$ as long as C-B. If we strike an arc G-H from A as center, with a radius $2/3$ of the distance C-B,

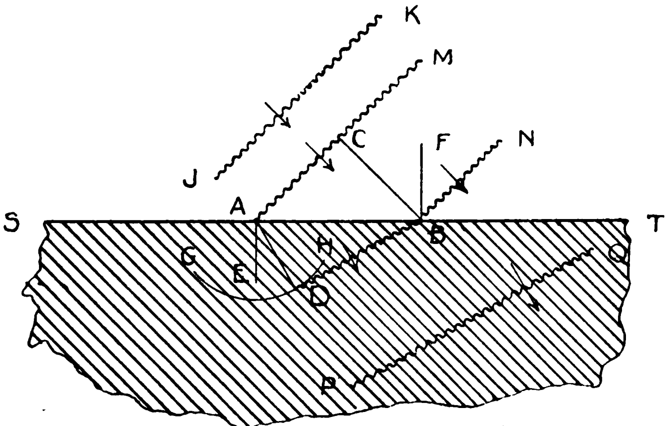


FIG. 1.

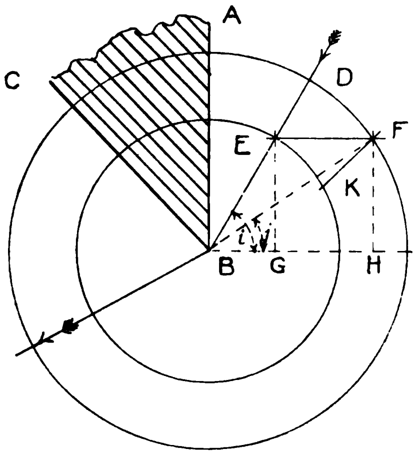


FIG. 2.

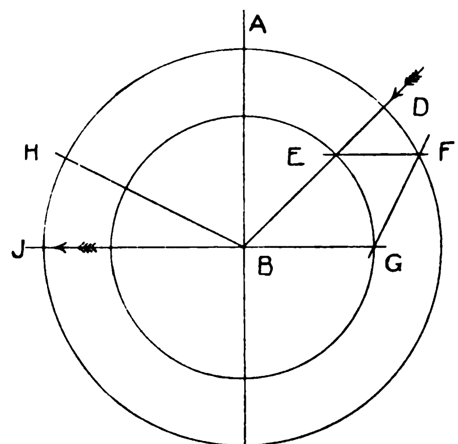


FIG. 3.

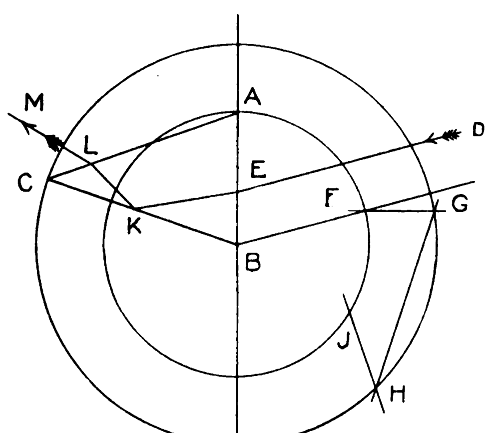


FIG. 4.

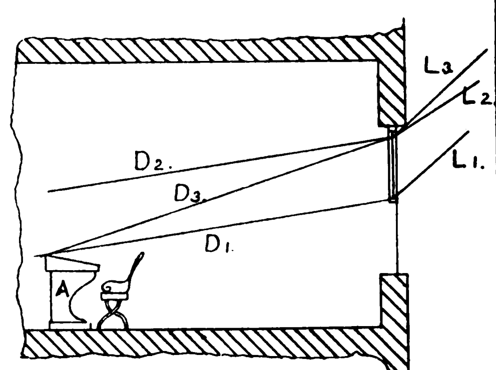


FIG. 5.

we are quite sure that the part of the wave formerly at A will be found somewhere upon this arc. The wave fronts J-K and A-M are practically straight lines and we know by experience that the wave fronts after passing through a plane surface are still straight lines. If we draw through B a tangent to the arc G-H this line B-D will fulfill all the conditions above for the wave front at the second instant of time considered, and by a more elaborate discussion may be shown to be the actual position of the wave front.

It will be noticed that the direction of the wave has changed. If v_1 is the velocity of light in air and v_2 is the velocity of light in glass,

$$\frac{v_1}{v_2} = \frac{CB}{AD} = \frac{AB \sin CAB}{AB \sin ABD} = \frac{\sin CBF}{\sin DAE} = \frac{\sin i}{\sin r}$$

where i is the angle of incidence CBF of the ray striking the glass and r is the angle of refraction DAE of the ray after entering the glass.

Above we defined the index of refraction as the ratio of the two velocities $\frac{v_1}{2}$. We now find that this index of refraction is equal to the ratio of the sines of incidence and refraction. This definition is the one which is used in all mathematical work in light and in what follows we shall use for our law of refraction, $n = \frac{\sin i}{\sin r}$.

THE PATHS OF RAYS THROUGH PRISMS.

Let ABC (Fig. 2) represent a section through a refracting prism, and let the light fall upon the prism in the direction DB. We wish to find the direction which the light takes in the prism. Construct at B, as center, two circles the ratio of whose radii is equal to the index of refraction. The path of the incident ray strikes the inner circle at E. Project E perpendicular to AB, striking the larger circle at F. FB is the direction of the ray in the prism.

For,

$$\sin i = \frac{EG}{EB}$$

$$\sin j = \frac{FH}{FB}$$

$$EG = FH$$

$$n = \frac{FB}{EB} = \frac{\sin i}{\sin j}$$

i is the angle of incidence, therefore j is the angle of refraction.

To find the direction in which the light leaves the prism, project F perpendicular to the face CB, striking the smaller circle at the point K. KB is the direction in which the light leaves the prism. The reasoning is similar to the above.

Let us find the prism angle required to change light from one given direction to another given direction. In Fig. 3, *AB* represents the surface of the prism. The light falls upon this surface in the direction *DB*, and we wish to throw this light in the direction *BJ*. Project *E* perpendicular to *AB*, striking the outer circle at *F*. Produce *JB*, striking the smaller circle at *G*. Draw *BH* perpendicular to *FG*. *ABH* is the prism angle required.

Fig. 4 shows the path of the ray *DE* through the reflecting prism *ABC*. *FB* is drawn parallel to *DE*. *F* is projected perpendicular to *AB*. *G* is projected perpendicular to the reflecting surface *BC*. *H* is projected perpendicular to the refracting surface *AC*. *JB* is the direction of the ray on leaving the prism.

PRISM CURVE TABLE.

Upon the prism curve table the curve marked *J* was constructed in the following manner: A diagram similar to Fig. 2 was made for a prism having an angle *ABC* equal to thirty degrees, and a large number of rays *DB* were drawn striking the prism, and the direction of each of these upon leaving the prism was ascertained. For instance, it was found that the light coming at an angle of eighty degrees from the vertical leaves the prism seven degrees above the horizontal; that a ray coming at an angle of seventy-four degrees from the vertical leaves the prism in a horizontal direction; that a ray striking at an angle of fifty-nine degrees from the vertical leaves the prism at an angle of fifteen degrees below the horizontal. These various points were plotted on squared paper similar to that used in the curve table and the smooth curve marked *J* on the prism curve table was drawn through these points. The curves for the other prisms were drawn in the same manner.

The numbers on the lower line of the prism curve table, 20, 30, 40, etc., are the zenith distances of the rays striking the prism plates. The numbers at the left side of the table are the directions in which the rays leave the prisms. *H* indicates the horizontal direction, the numbers below *H* indicate degrees below the horizontal, and the numbers above *H* indicate degrees above the horizontal. For instance; a ray striking the fifty degree prism at an angle of sixty degrees from the vertical, leaves the prism in a horizontal direction. The ray striking this prism at an angle of sixty-five degrees from the vertical leaves seven degrees above the horizontal, and the ray striking this prism at an angle of fifty degrees from the vertical leaves at an angle of eleven degrees below the horizontal. The prism curve table saves us the trouble of making a diagram such as Fig. 3, whenever we wish to change light from one direction to another. For instance; if we wish to throw light coming from an angle of fifty-five degrees from the vertical into a direction one degree below the horizontal, we see at once that the fifty-five degree prism will answer this purpose.

At the right of the prism curve table is a table made in the following manner: The headings of the columns 10, 20, 30, etc., mean feet distant from the window inside the room. The numbers given in the columns below are the product of the number at the head of the column multiplied by the tangent of the angle given at the left of the table. This is convenient in estimating the direction in which we wish light thrown into the room. If we wish the light thrown toward a point which is one hundred feet from the window and seven feet below the window, we look for the 100 and 7 at the right side of the table, instead of reducing this to degrees and looking at the left side of the table.

TO DETERMINE WHAT PRISMS TO PLACE IN A WINDOW.

The general method of lighting by means of Luxfer Prisms is to throw the light from the sky directly into the room. We treat the sky itself as a luminous surface. We do not derive our light from the sun directly. If we did this, we should need to change our prisms every hour of the day. The sky itself is very bright in comparison with ordinary objects. The light from the sky ordinarily falls upon a window and goes straight through and reaches the floor at a point not very far distant from the window. The floor is of a dark color, reflecting perhaps only one-tenth part of the light falling thereon, so that perhaps nine-tenths of the light is lost upon reaching the floor. If prisms are placed in the window the light is thrown directly back into the room before striking the floor. The first object which this light strikes is the one which we wish to illuminate, and in general it is of a light color. The prism plate in the window practically takes the place of a skylight. It is to be thoroughly understood that prism plates placed in windows do not increase the quantity of light entering the room. They simply re-distribute the light in such a manner that it is utilized to a much better advantage.

UNIFORM PLATE.

In determining the proper prism to place in any window, we must know both the direction in which the light falls upon the prism, and the direction in which we wish to have the light leave the prism. If we know the direction in which the lowest light falls upon the prism, we know for ordinary cases that the other directions lie between this and the vertical, and if we know the direction in which we want the highest light thrown in the room, the other directions in which we wish light will be between this and the vertical in a downward direction. If we throw the light which falls upon the prisms in the lowest direction so that it leaves the prisms in the highest direction desired, then, in most cases, the other rays of light falling upon the prisms between the lowest direction and the vertical will illuminate other parts of the room which we wish lighted, and our problem is substantially solved.

In Fig. 5, let L_1 be the lowest direction in which light falls upon the prism plate, and let A be a desk which it is desired to illuminate and which is the object farthest back in the room and highest up from the floor that we wish to illuminate. If such a prism is placed in the window that the light L_1 (striking the lowest prism in the window), takes the direction D_1 (the direction from the lowest prism to the desk), then the desk A will be lighted by every prism in the window; for, although the light L_2 which is parallel to L_1 and strikes the upper prism of the window, goes above the desk A, there will be some other ray L_3 higher than L_2 which will take the direction D_3 and strike the desk. All the other rays above L_3 will go lower than D_3 and into the part of the room which we wish to illuminate, and practically all the light striking the window will be utilized.

If there is no prism which will throw the lowest ray directly upon the point desired, it is usually best to select the one which throws the light above the point. In rough work it is sufficient to throw the light in a horizontal direction and in rooms which require a general illumination it is better to throw the light rather high than to confine it too nearly to the floor. For the general effect of the window plate upon people inside the room, it is commonly best if there is light enough to spare, to arrange the prisms so that none of them will appear dark to a person in any part of the room.

PLATE OF VARYING PRISMS.

In may happen in case the prism plate is quite deep and in case the desk A is quite near the window, that it will be desirable to place a prism in the top of the plate which is different from that placed in the bottom of the plate. Suppose that we wish to place prisms in an upper sash five feet deep, the zenith distance being fifty degrees, and we wish to light a desk twenty feet from the window, the top of which is four feet below the lower edge of the upper sash. Looking at the right of our table, finding the column headed twenty, and taking the line marked 3.9, and following it to the left we find it intersects the vertical line marked fifty upon the L prism curve. If, therefore, we place the L prism in this window we will light the desk satisfactorily, and the most of the remaining light will fall between the desk and the window. It may be noticed, however, that the K prism curve crosses the fifty degree angle at 6.5 feet below the horizontal, twenty feet back. So that, if we put L prisms in the lower half of our window and K prisms in the upper half, we shall accomplish the same result so far as the desk is concerned, and shall not throw so much light over the desk, but will save this for the space between the desk and the window, the area which we wish to illuminate.

LIGHT NEAR THE WINDOW.

Referring now to the intensity curve table, it will be noticed that the space which before was brightest in a room is now left practically without light. The lowest light from the P prism is 23° below the horizontal; from the N prism 31°, etc. It is evident, therefore, that there is a dark area in a room near the window. In order to obviate this difficulty, it is absolutely necessary in almost all cases to insert a few prisms of low angle in the prism plate. For instance: given a zenith distance of 40° we should probably choose either the 65° or 70° prism for the body of the prism plate. Suppose it is 65°. The light from this prism goes down as low as 28° below the horizontal, but for the last few degrees the light is not very brilliant, as indicated by the intensity curve to be explained later. In order to illuminate the area left dark, it will be necessary to insert either some J or K prism lights. The quantity of the low angle prisms can be obtained roughly by considering the area to be lighted by these and by the body of the prism plate. If the area to be lighted by the body of the prism plate is on the average 30 feet from the window, and if the area to be lighted by the low angle prisms is on the average 10 feet from the window, then, since roughly the intensity of the light varies inversely as the square of the distance from the window, the proportion of the two window lights should be about as 100 is to 900. The proportion ordinarily used is from 10 to 20 per cent. As a matter of choice in the case illustrated above between the J and the K prisms, probably the K should be selected simply as a matter of appearance. If the J prism were selected it would be possible to select a point in the room such that there would be a line of prisms across the window, all of which would be dark.

The method of using the intensity curve table rapidly is to select the prism, trace this curve down to the left hand side and then trace the horizontal line to the right back to the zenith distance and select the proper prism curve at that point for the low angle prism light. In many cases even where the light is not required up close to the window, it is still advisable to insert a few of these prism lights of low angle in order to relieve the monotony of the uniform prism plate. This effect is seen both from the inside and from the outside.

RAYS NOT IN THE PRINCIPAL PLANE.

EDGE.

The edge of a prism is the line in which the two refracting faces intersect.

PRINCIPAL PLANE.

The principal plane of a prism is the plane which is perpendicular to this edge. Although we have not said anything heretofore about the principal plane, it has been tacitly understood that we have been treating of rays which are in this plane. Other rays have not been entirely disregarded, but it has been assumed that if the rays which pass through the prism in the

Fig. 6

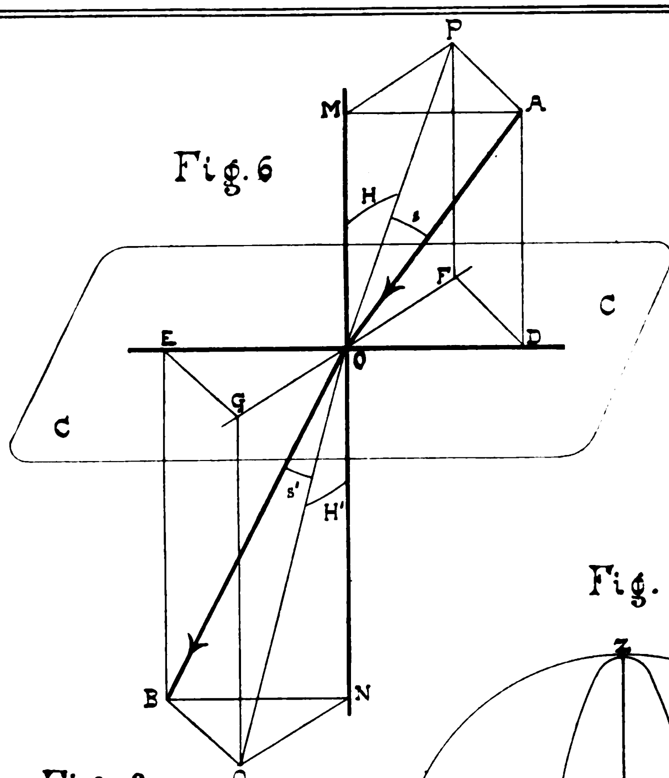
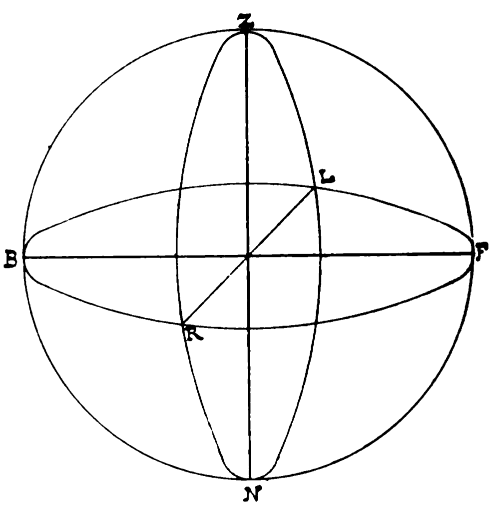


Fig. 7

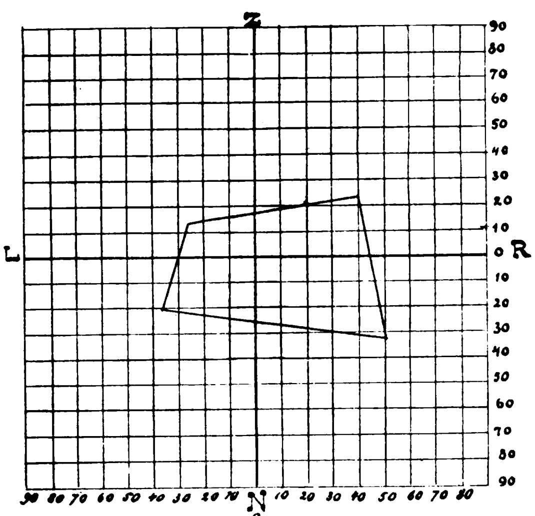


FLBR = Horizontal Plane

ZLNR = Wall Plane

ZFNB = Primary Plane.

Fig. 8



principal plane are attended to carefully, the side rays will take care of themselves. As a matter of experience, we know that in the case of a uniform building across the street, if the prisms throw the direct light high enough no trouble is ever found with the side light. Indeed, we shall soon see that the side light goes higher than the direct light. It often happens that by the side of the building across the street stands one a few degrees higher. We should know which building requires the higher prism. It often happens that this second building is so high that tilted prisms are necessary. We should be able to find the final direction of any ray from such a sky. We shall take up, first, refraction at a single surface, finding two general truths, which we shall then apply to the prism. We shall find a fairly convenient and accurate method of working out all information needed in actual work.

PLANE SURFACE.

In Fig. 6, CC represents the upper surface of a body of glass. The ray AO strikes this surface at O and gives rise to the refracted ray OB. We shall suppose that AO is drawn of unit length and OB of a length equal to the index of refraction, n. MON is a normal to the surface at O. Upon this drop the perpendiculars AM and BN. These lines AM and BN are equal and parallel. All these lines are in the plane $MODE$. Let us now consider some other plane passing through the normal such as $MPFG$. Upon this plane drop the perpendiculars AP and BQ, and draw the lines MP, PO, NQ, QO. The angles PMA and QNB are equal and the triangles PMA and QNB are equal in all respects. If $s=AOP$, $H=POM$, $s'=BOQ$, $H'=QON$, remembering that AO is unity and OB, is n, we have

$$\begin{cases} \sin s = PA = QB = \text{n} \sin s', \\ \cos s \,.\, \sin H = PM = QN = \text{n} \,.\, \cos s' \,.\, \sin H'. \end{cases}$$

1. $\sin s = \text{n} \sin s'$.

2. $\sin H = \text{n} \,.\, \dfrac{\cos s'}{\cos s} \,.\, \sin H'$.

These are two very important principles.

A. *The angles which the incident and refracted rays make with any plane through the normal to the refracting surface are related by the law of refraction.*

B. *The projections of the angles of incidence and refraction upon any plane through the normal to the refracting surface are related by the law of refraction, the index of refraction being changed to* $n \,.\, \dfrac{\cos s'}{\cos s}$.

PRISM.

The application of these two principles to prisms is easy. At the first face of a prism let s be the angle between the principal plane and the ray when outside the glass and s' the angle when inside. Let l and l' be the corresponding angles at the second face of the prism. Now it is to be noticed that both s' and l' are angles between the principal plane and the ray inside the glass. They both refer to the same ray and to parallel principal planes so that they must be equal.

$$l' = s'.$$
$$\sin s = n \sin s'$$
$$\sin l = n \sin l'.$$
$$\therefore \quad l = s.$$

If the prism were a reflecting prism there will follow one more step in the proof, but if the principal plane remains the same the following result is independent of the number of internal reflections.

C. *The angle between a ray and the principal plane is always the same in the same medium.* The angle between a ray and the principal plane is the same upon leaving the prism as it was upon entering it.

Corresponding to (B) above, we have for the prism:

D. *The projections of the incident, refracted, reflected, and emergent rays upon the principal plane of a prism obey the same laws as rays in the principal plane, the index of refraction being changed to* $n \cdot \dfrac{\cos s'}{\cos s}$.

Figures similar to Figs. 2, 3 and 4, may be used in tracing these projections of rays through prisms. It will be noticed that C and D furnish us means of finding the final direction of a ray striking any prism in any direction.

The following is a table of modified indices of refraction:

s	$n \dfrac{\cos s'}{\cos s}$	s	$n \dfrac{\cos s'}{\cos s}$
0°	1.53	45°	1.92
5°	1.53	50°	2.06
10°	1.54	55°	2.25
15°	1.56	60°	2.52
20°	1.59	65°	2.92
25°	1.62	70°	3.53
30°	1.67	75°	4 58
35°	1.73	80°	6.74
40°	1.81	85°	12.96

These may be obtained graphically by means of a figure similar to Fig. 2. The side angle s is represented by EBH. If BE is unity and if BF is n, then the modified index of refraction is represented by the distance from B to the point of intersection of BE and HF.

SKY GLOBE.

In Fig. 7, *BLFR* represents the HORIZONTAL PLANE. *ZLNR* represents a plane passing through the vertical wall upon which a prism plate is mounted. We shall call this the WALL PLANE. *ZFNB* represents a vertical plane which is at right angles to the wall plane. We shall call this the PRIMARY PLANE. We shall call

Z, Zenith point.
N, Nadir point.
F, front point.
B, back point.
R, right point.
L, left point.

If we obtain a small globe which has been painted black, we may draw upon it the three great circles described above, their intersections corresponding to the points above defined.

SKY DIAGRAM.

The vertical edge of a building is represented upon the sky-globe by an arc of a great circle passing through the zenith and nadir points (*Z* and *N*). The top of an uniform building which is parallel with the street (*LR*) is represented by an arc of a great circle passing through the right and left points. If this building makes an angle (b) with the street the great circle representing its top will pass through two points upon the horizon line at a distance (b) from the right and left points. In particular if the building is at right angles to the street, $b=\frac{\pi}{2}$, and the great circle passes through the *front* and *back* points (*F* and *B*). We are now able to draw the outline of any sky, the sky of course is represented by the area inclosed.

STRAIGHT PRISM PLATE.

In the case of the ordinary vertical prism plates, the lights of which are not tilted, it is noticed that the principal plane coincides with the primary plane. We draw our sky diagram as above. Select one corner and let us find in what direction a ray from this corner emerges from a given prism. Drop a perpendicular from the corner to the principal plane. The length s of this perpendicular shows, first, that the ray emerges at the same distance from the principal plane, but upon the other side; it shows, secondly, the index of refraction to use. Taking this index of refraction, we work out by diagram the point on the principal plane which is the projection of the emergent ray. The real point of emergence is at a distance s from this point as explained above.

Treat each corner of the sky diagram as above and connect in order the points of emergence thus found; the area thus inclosed represents the solid angle lighted by the window. We may map this solid angle very conveniently, using the point B as origin, as shown in Fig. 8 (using Mercator's projection). Lines parallel to LR represent great circles on the sky globe, and those parallel to NZ, small circles having L and R as centers. The points obtained may be conveniently recorded by these co-ordinates, distances along BZ and BR being positive.

TILTED PRISMS.

When the building directly across the street is very high, with an adjoining building quite low, it is often advisable to tilt the prism lights in the plane of the prism plate. In working out the solid angle of light for a window where these tilted lights are used it must be carefully remembered that our principal plane no longer coincides with the primary plane, but the two intersect at the front and rear points, making an angle equal to the angle of tilt. The projections of the corners of sky must be made upon the principal plane, not the primary plane, and in interpreting the results care must be used not to mix the two planes. Such a prism and such an angle of tilt must be selected as gives the best distribution of light in the room.

CANOPY.

When we use the straight canopy, set at a slope a from the vertical, we return in our sky globe to the old principal plane—the primary plane, but the plane representing the prism plate is no longer the wall plane, but intersects this at the right and left points at an angle a. Measure angles of projection points from surface of canopy—not from zenith point.

TILTED CANOPY.

An additional complication arises when we use tilted canopy lights. We have as before our prism plate plane making an angle a with the wall plane. The normal to the prism plate is in the primary plane, and at a distance $(\frac{\pi}{2} - a)$ from the zenith and nadir points. The principal plane of the tilted prisms must pass through this normal and hence is a great circle, intersecting the primary plane at a distance $(\frac{\pi}{2} - a)$ from the zenith and nadir points, and makes an angle with the principal plane equal to the angle of tilt (T). Care must be used in working the rays through the sky globe and interpreting the results.

QUANTITY OF LIGHT FALLING UPON A PRISM PLATE.

If we imagine a prism plate at the center of a sphere and imagine it illuminated through a small hole in the surface of the sphere the illumination due to the direct light is proportional, (1), to the brightness of the luminous surface held before the small hole; (2), to the area of the hole; (3), to the reciprocal of the square of the radius of the sphere; (4), to the area of the prism plate; and (5th), to the sine of the angle between the prism plate and the radius touching the hole. If the hole subtends at the center a small solid angle $d\Omega$, the radius becomes unity and we may write the expression $\boldsymbol{B} WD \sin t.\ d\Omega$.

If the hole in our sphere is of considerable size we must integrate the above expression over the entire solid angle subtended by the hole.

$$\mathrm{N} = WD\int_{o}^{\Omega} \boldsymbol{B} \sin \mathrm{t}\ \mathrm{d}\Omega.$$

The integral is the illumination at the surface of the prism plate and the product of this by the area in square inches, as above, we have called the Numeral, indicated by N.

In our work upon the sky we shall assume that $\boldsymbol{B}$, the brightness of the sky is constant, and we shall omit this factor for the present. In Fig. 9, a prism plate is represented by P set in a vertical position. The right half of the sky is represented by the surface ZRF. In order to express the position of any point (B) upon the sky, we pass through this point two great circles AF and CR, which also pass through the front and right points respectively. the primary angle ZC, and the wall angle ZA determine the point. We shall use p to denote primary angle and w to denote wall angle. For present purposes we shall let p_o denote zenith distance of a building across the street and w_o the wall angle of a side building, adding r and l to indicate which side. We shall use s to denote the distance BC for any point. s_r the same for any point along the *limiting* line AF and $s_o - o$ for the point $p_o\ w_o$

If DC is dp, GB is $dp.\ \cos s$, BJ is ds, and the element of area (or of solid angle) of sky $BGHJ$ is $\cos s.\ dp.\ ds$. The angle (t) between the surface of the prism plate and the radius touching B is the angle BA. From spherical trigonometry we know that in the spherical triangle BAR right angled at A,

$$\begin{aligned}\sin \mathrm{AB} &= \sin RB' \sin ARB \\ &= \cos\ s.\ \sin p.\end{aligned}$$

We write now for our numeral,

$$\begin{aligned}\mathrm{N} &= WD\int \sin \mathrm{t}\ .\ \mathrm{d}\Omega \\ &= WD\int\int \cos s\ .\ \sin p\ .\ \cos s\ .\ \mathrm{d}p\ .\ \mathrm{d}s \\ &= WD\int\int \sin p\ .\ \cos^2 s\ .\ \mathrm{d}p.\ \mathrm{d}s.\end{aligned}$$

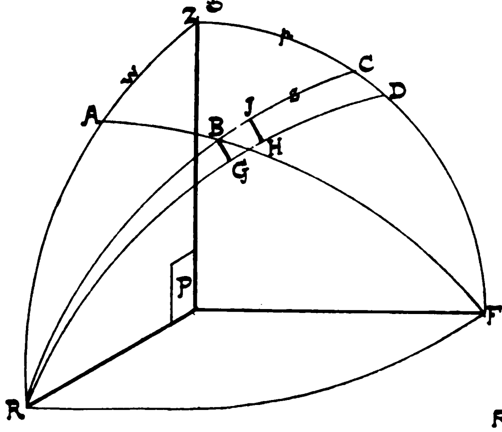


Fig. 9

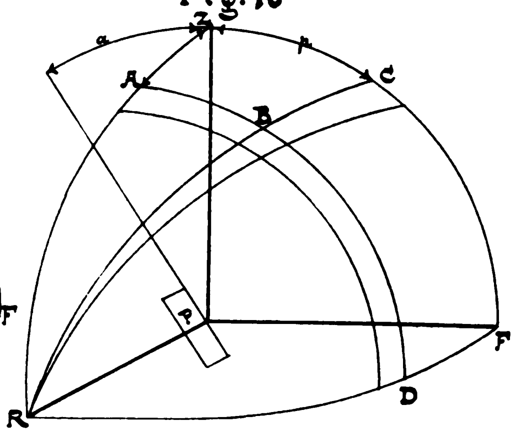


Fig. 10

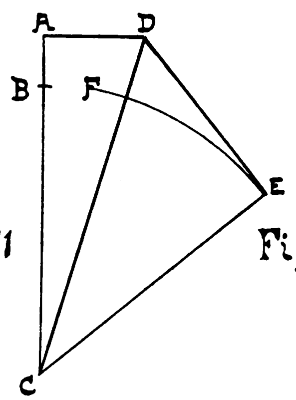


Fig. 11

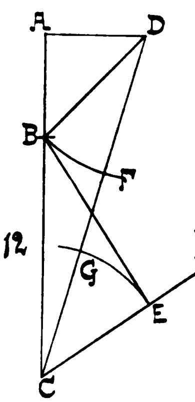


Fig. 12

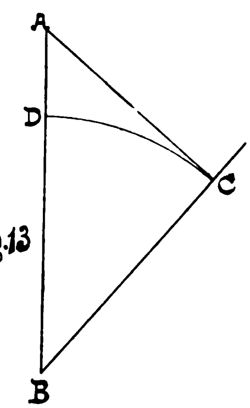


Fig. 13

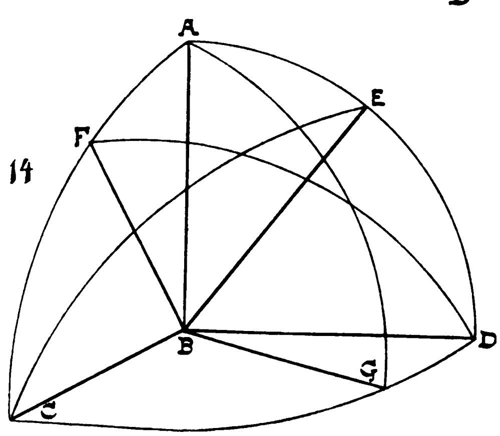


Fig. 14

If we integrate this with respect to s between the limits o and s_r, our element becomes the area of a part of a lune *CBGD*, all in the right half of the sky. If we then integrate with respect to p, between the limits o and p_o, the result is the numeral for the right half of the whole sky exposed. The numeral for the left half will be of exactly the same form and may be added when we get through.

$$N_r = WD\int_o^{po}\int_o^{sr} \sin p \,.\, \cos^2 s \,.\, dp.\, ds.$$

$$= WD\int_o^{p_o} \sin p \left[\tfrac{1}{2}\cos s \sin s + \tfrac{1}{2} s\right]_o^{s_r} .\, dp$$

$$N_r = \tfrac{1}{2}\, WD \,.\int_o^{p} \sin p \left(\cos .\, s_r \,.\, \sin s_r + s_r\right) dp.$$

Now s_r is a function of p, but we may just as well regard p as a function of s_r. When $p = o$, $s_r = w_o$, when $p = p_o$, $s_r = s_o$; and in general we know from spherical trigonometry that in the spherical triangle *BCF*, right angled at *C*,

$$\sin CF = \tan BC \,.\, \cot BFC,$$
$$\text{or } \cos p = \tan s_e \,.\, \cot w_o .$$

Differentiating

$$-\sin p \,.\, dp. = \cot w_o \,.\, d\,(\tan s_r).$$
$$= \cot w_o \ \sec^2 s_r \ d\, s_r .$$

Substituting

$$N_r = -\tfrac{1}{2}\, WD\int_{w_o}^{s_o} \cot w_o \left(\sec^2 s_r \,.\, \cos s_r \,.\, \sin s_r \,.\, d\, s_r + s_r \,.\, d \tan s_r\right)$$

$$= -\tfrac{1}{2}\, WD\int_{w_o}^{s_o} \cot w_o \left(\tan s_r\, d\, s_r + s_r\, d \tan s_r\right)$$

$$= -\tfrac{1}{2}\, WD\Big[\cot w_o \,.\, s_r \tan .\, s_r\Big]_{w_o}^{s_o}$$

$$= \tfrac{1}{2}\, WD\Big[\cot w_o \,.\, w_o \,.\, \tan w_o - \cot w_o \,.\, s_o \,.\, \tan s_o\Big]$$

$$= \tfrac{1}{2}\, WD\Big[w_o - s_o \,.\, \tan s_o \,.\, \cot w_o\Big].$$

We have seen above that

$$\cos p_o = \tan s_o \, . \, \cot w_o$$
$$\tan s_o = \cos p_o \, . \, \tan w_o \, .$$

Substituting

$$N_r = \tfrac{1}{2} WD \Big[w_o - \cot w_o \, . \, \cos p_o \, . \, \tan w_o \, . \, \tan^{-1} \Big(\cos p_o \, . \, \tan w_o \Big) \Big] .$$

$$= \tfrac{1}{2} WD \Big[w_o - \cos p_o \, . \, \tan^{-1} \Big(\cos p_o \, . \, \tan w_o \Big) \Big] .$$

If $w_o = \frac{\pi}{2}$

$$N_r = \tfrac{1}{2} WD \Big[\frac{\pi}{2} - \frac{\pi}{2} \cos p_o \Big]$$

$$= \frac{\pi}{4} WD \Big[1 - \cos p_o \Big] .$$

Of course the whole numeral is obtained by adding that for the left side of the sky. It is interesting to see that our final result for an open sky, with the exception of a constant factor, is identical with the First Numeral, which has been in use for some time.

THIRD NUMERAL.

Fig. 10 represents a prism plate P, set at a slope (a) from the vertical. Any point B upon the sky is found by passing through it the great circle RBC and the small circle ABD [pole at R], the co-ordinates are ZC and ZA, indicated by p and s respectively. For our element of sky area we have

$$d\Omega = \cos s \, . \, ds \; dp.$$

$$\sin t = \cos s \, . \, \sin (a+p).$$

$$N = WD \int \sin t \, . \, d\Omega$$

$$= WD \int \int \cos^2 s \, . \, \sin (a+p) \, ds. \, dp.$$

$$N = WD \int_{s_l}^{s_r} \cos^2 s \, . \int_{o}^{p_o} \sin (a+p) \, dp.$$

$$= \frac{WD}{2} \Big[\sin s_r \cos s_r - \sin s_l \cos s_l + s_r - s_l \Big] \Big[\cos a - \cos (a+p) \Big] .$$

If $s_l = - s_r = s$

$$N = WD \Big[\sin s \cos s + s \Big] \Big[\cos a - \cos (a+p) \Big] .$$

If $s=\frac{\pi}{2}$

$$N=WD\,.\,\frac{\pi}{2}\left[\cos a-\cos\,(a+p)\right].$$

If $a=o$.

$$N=WD\,.\,\frac{\pi}{2}.\left[1-\cos p\right].$$

If in the above $p=32°$

$$N=WD\Big(1-\cos\,(a+p)\Big).$$

If $a=o$.

$$N=WD\Big(1-\cos p\Big).$$

In actual practice not much light is utilized in a room through a greater horizontal range than 60° no matter whether the sky is open at the side or not. Making this assumption we may adopt the second equation above as our definition of the First Numeral.

FIRST NUMERAL.

Fig. 10 shows a section through a window in which is placed the prism plate of width W and depth D. The reveal immediately overhanging the window is of width r. The zenith distance is z, which is constant for the window, since the buildings limiting the horizon are distant. Let x be the distance of any point on the window below the edge of the reveal, and let b be the zenith distance of the reveal at that point on the window. Accordingly,

$$N_1=W\int(\cos b-\cos\,z)\,dx \qquad - \; - \; - \; - \; - \; - \qquad 1.$$

Substituting for b its value in terms of x and r and introducing the proper limits of integration, our First Numeral becomes,

$$N_1=W\int_{r\cot z}^{D}(\cos\,(\tan^{-1}\frac{r}{x})-\cos z)\;dx \qquad - \; - \; - \; - \; - \; - \qquad 2.$$

$$N_1=W\int_{r\cot z}^{D}(\frac{x}{\sqrt{x^2+r^2}}-\cos z)\;dx \qquad - \; - \; - \; - \; - \; - \qquad 3.$$

which when integrated becomes,

$$N_1=W\Big[\,(\sqrt{x^2+r^2}-x\cos z)\,\Big]_{r\cot z}^{D} \qquad - \; - \; - \; - \; - \; - \qquad 4.$$

Substituting the limits of integration we have

$$N_1 = W(\sqrt{D^2 + r^2} - D \cos z) - Wr\,(\sqrt{\cot^2 z + 1} - \cot z \cos z) \quad 5.$$

This expression when simplified takes the form

$$N_1 = W(\sqrt{D^2 + r^2} - D \cos z - r \sin z) \quad - \quad - \quad - \quad - \quad 6.$$

This is our final expression for the First Numeral in the case of the window shown in the figure. If we suppose that the reveal is at a height h above the top of the prism plate, the formula No. 6 is slightly changed and there arises two cases. The first case is where the upper edge of the prism plate is entirely in the shade, *i. e.*, when h is less than $r \cot z$. The formula in this case is

$$N_1 = W\left[\sqrt{(D + h)^2 + r^2} - (D + h) \cos z - r \sin z\right] \quad - \quad 7.$$

If the upper edge of the prism plate is not in the shade, *i. e.*, if h is larger than $r \cot z$, we must change the limits of integration in formula No. 4, the lower limit being h instead of $r \cot z$ and the upper limit being $D+h$ instead of D. The formula when simplified takes the form,

$$N_1 = W(\sqrt{h + D)^2 + r^2} - \sqrt{h^2 + r^2} - D \cos z) \quad - \quad - \quad 8.$$

In case we have no reveal, $r=o$, and this formula is reduced to the original formula,

$$N_1 = WD(1 - \cos z) \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad 9.$$

It will be noticed in formula No. 7 that the first term in the parenthesis is the distance from the outer edge of the reveal to the lower edge of the prism plate. The sum of the second and third terms is the projection of this line upon the lowest incident ray of light. The various terms in formulæ Nos. 8 and 9 are interpreted in a similar manner. It is evident, therefore, that the First Numeral may be obtained graphically in a very simple manner. Figs. 11, 12 and 13 show the construction for formulæ Nos. 7, 8 and 9 respectively. In Fig. 11, BC is the depth of the prism plate; AB is the height h; AD is the reveal r; ACE is the zenith distance. In order to get the parenthesis factor of equation 7, draw DC, project D perpendicular to CE; strike the arc EF with C as center. DF is the required parenthesis factor. In Fig. 12, FG is the parenthesis factor of equation No. 8, and in Fig. 13, AD is the parenthesis factor of equation No. 9.

The above expression, as has been pointed out, is proportional to the quantity of light striking a window, the limitations as given above, being carefully attended to. For various reasons this expression is not proportional to the illumination given by various windows. In the first place the walls of the room have a great deal to do with the illumination, and this question we shall consider later, more in detail. The illumination is not proportional to the numerals for various prisms because the prisms distribute the light differently from the front, back in the room. If prisms were inserted as is recommended on page 9, so that the light falling upon the prism plate of high angle is distributed from the front to the rear of the room in a manner similar to the distribution given by a lower prism angle, then, with the walls and other conditions the same, the illumination given by prisms of low and high angles would be strictly comparable. As we do not ordinarily insert in plates of high prism angles, enough prism lights to illuminate the front part of the store to the same degree as the rear in comparison with the light given by smaller prisms, the numeral should be too small for high prism angles and somewhat too large for low prism angles.

It must be carefully noticed that this first numeral is not applicable to light wells, where the horizon is limited at the sides, nor to cases of a very irregular horizon.

DIRECT ILLUMINATION AT ANY POINT IN THE ROOM.

VERTICAL DENSITY.

We have seen that the quantity of light striking a prism plate between limiting angles of incidence e and f is proportional to the difference of the sines of e and f. We shall obtain now an expression for the vertical density of the rays of light upon leaving the prism, *i. e.*, the quantity of light in any small vertical angle divided by the angle for unit depth of the prism plate. Assume that we have a prism of angle i, the light striking this at an angle of incidence u and leaving the prism at an angle y with the normal. The angles of refraction corresponding to u and y, are v and w respectively. If we let $x = \sin u$, then the expression for the vertical density of the light leaving the prism is $D = -\frac{dx}{dy}.$ This vertical density D when multiplied by the secant of the angle of *dip* is proportional to the apparent brightness of the prism plate viewed from the inside of the room. We suppose, of course, that the sky is of uniform brightness. We shall derive an expression for the value of D in terms of the prism angle and the direction in which the light leaves the prism. Let n be the index of refraction of the prism and let p be the reciprocal of n. We have then,

$$\left.\begin{array}{l}\sin u = n \sin v \\ \sin y = n \sin w\end{array}\right\} \qquad 1.$$

$$v + w = i \qquad 2.$$

We shall introduce $\sin u = x$ as a new variable, and shall get an expression for x involving only i and y and differentiate this with respect to y, thus giving us D.

$\sin v = px$ - - - - - - - - - - 3.

$\sin w = p \sin y$ - - - - - - - - - 4.

$$\left.\begin{array}{l}\sin^{-1}(px) + \sin^{-1}(p \sin y) = i \\ \sin^{-1}(px) = i - \sin^{-1}(p \sin y)\end{array}\right\} \qquad 5.$$

$px = \sin(i - \sin^{-1}(p \sin y))$ - - - - - - - 6.

Let $j = \sin^{-1}(p \sin y)$

$\sin j = p \sin y$

$\cos j = \sqrt{1 - p^2 \sin^2 y}$

$x = n \sin (i - j)$

$= n (\sin i \sqrt{1 - p^2 \sin^2 y} - p \cos i \sin y)$ - - - - 7.

Now differentiating No. 7 with respect to y, we have the value for $\boldsymbol{D}$.

$$\boldsymbol{D} = -\frac{dx}{dy} = \sin i \frac{\sin y \cos y}{\sqrt{n^2 - \sin^2 y}} + \cos i \cos y \qquad 8.$$

If we wish to obtain the density along the horizontal direction, place $y = i$, substituting this in the above we have,

$$D_h = \frac{\sin^2 i \cos i}{\sqrt{n^2 - \sin^2 i}} + \cos^2 i \qquad 9.$$

We wish to find the illumination at a point P (Fig. 16) inside a room illuminated by a window prism plate having prisms of angle i. The plate is of width W and is at a distance S from the point. The line from the lower edge of the plate to the point makes an angle $(i - y_2)$ with the horizontal, while the angle to the upper edge of the plate is $(i - y_1)$.

ILLUMINATION.

If D is the density of the rays in any direction, and C a constant to be determined by experiment, the illumination at the point is equal to the following expression:

$$L = C \frac{W}{S} \int_{y_1}^{y_2} D \, dy \qquad 1.$$

$$L = C \frac{W}{S} \int_{y_1}^{y_2} \left(\frac{\sin i \sin y}{\sqrt{n^2 - \sin^2 y}} + \cos i \right) \cos y \, dy \qquad 2.$$

When this is integrated and reduced we have the following as the illumination at the point, due to the window prism plate directly:

$$L = C \frac{W}{S} \Big[\cos i (\sin y_2 - \sin y_1) + \sin i (\sqrt{n^2 - \sin^2 y_1} - \sqrt{n^2 - \sin^2 y_2}) \Big] \qquad 3.$$

This expression will enable one to estimate the quantity of prisms needed to illuminate a desk if it is known that a person using the desk ordinarily uses a light of a certain candle power in a certain position. It is to be understood, of course, that this gives only the direct light and that the light which arises by diffusion from the other parts of the room will materially increase this illumination, and in making any such estimate, this extra light will need to be considered very carefully.

A PROVISIONAL LIST OF PHOTOMETRIC UNITS,

BY HENRY CREW.

The insuperable difficulty of measuring photometric quantities in mechanical units renders more or less unsatisfactory any system of units we may adopt for dealing with luminous energy.

For the sake, however, of intelligent communication with each other, some such system is indispensable.

We have accordingly adopted the following as representing the best scientific usage.

In practice we shall seldom, if ever, have occasion to deal with the total radiation which any source of energy emits.

For the present we are engaged in transmitting only that portion of the total radiant energy which is capable of affecting the retina of the normal eye.

To this fraction of the total radiant energy we shall give the name,

LUMINOUS ENERGY.

Concerning this term the following two points are to be borne in mind: 1st, that while it is a practical impossibility to go into the laboratory and measure just what fraction of the total radiant energy exists in the form of luminous energy, yet this fraction is a *perfectly definite* quantity; 2nd, "Luminous energy" is equivalent to "light" only when the latter is used in the narrow sense so as *not* to include actinic and thermal effects.

These conventions fixed, we are ready to consider the following photometric quantities:

I. *Intensity* of a point-source, (or of a source which is sufficiently small compared with its distance to be treated as a point-source) is defined as *the amount of luminous energy emitted per second.*

The word "intensity" is used in a great variety of senses in scientific terminology. If, therefore, any ambiguity should at any time arise as to its exact meaning, it may be modified to read "luminous intensity" which is never employed in any sense except as above defined.

Concerning the nature of intensity in general, it need only be added that it does *not* represent a quantity of energy such as that contained in a storage cell or in a coiled spring. It is a *rate* of flow of energy, a ratio between a quantity of energy and a time. The product *intensity* x *time* is luminous energy and this product determines the amount of one's gas or electric-light bill.

Since it is impracticable to determine intensity in mechanical measure, the following unit is adopted:

Unit of Intensity is defined as *the intensity of a Hefner lamp in a horizontal direction, the dimensions* of the Hefner lamp being those prescribed by the Reichsanstalt at Berlin.*

Name: This unit is called "one candle."

Symbol for $\begin{cases} \text{intensity, J.} \\ \text{candle power, c. p.} \end{cases}$

II. *Luminous Current* is defined as *the rate at which luminous energy is emitted by a point-source through a solid angle of one steradian.*

Unit: The luminous current of one candle, *i. e.*, of one Hefner lamp.

Name: "Lumen." Proposed by L. Weber.

Symbol: for $\begin{cases} \text{luminous current, } \varphi \\ \text{lumen, } lm \end{cases}$

It is evident that, if a point-source radiated uniformly in all directions, its intensity would be 4π times its luminous current, *i. e.*

$$J = 4\pi\varphi$$

III. *Illumination* is defined as *the ratio of the luminous current to the area upon which it falls.*

This is the same as saying that the illumination is measured by the number of lumens per square centimeter at the point in question. The numerical value of the illumination at any point in a room measures, in general, the success with which that part of the room is lighted.

It must not be forgotten that of two equal illuminations, one produced by rays from one direction only, the other by rays from many directions, the latter is as a rule much more effective.

Illumination is a property of a surface at a point; and is determined only by the area and the light immediately incident upon it, independently of the source.

It is evident, however, that in case of a *point-source* the illumination varies inversely as the square of the distance between the point and surface; in case of a *linear source* the illumination varies inversely as the distance; in case of a *plane source, of practically infinite extent*, such as the sky, the illumination is entirely independent of the distance separating the source and the illuminated surface.

Unit of Illumination is one lumen per square centimeter.

Name: "Lux."

Symbol: for $\begin{cases} \text{illumination, E} \\ \text{Lux, } lx \end{cases}$

IV. *Brightness* is defined as *the luminous current leaving unit area of apparent surface.*

*For these dimensions, see Palaz, *Industrial Photometry*, pp. 136-143.

The fundamental distinction between *brightness* and *illumination* is that, in the former, the surface is considered as the origin of a luminous current; while in the latter, the surface is considered as the recipient of the luminous current.

Unit of Brightness, is *that brightness which yields one lumen per square centimeter of apparent surface.*

Name: Lumen per square centimeter.

Symbol: for $\begin{cases} \text{brightness, B} \\ \text{unit of brightness, } lm \text{ } per \text{ } cm^2 \end{cases}$

V. *Quantity of Light* is defined as the product of the luminous current by the time it flows.

Unit quantity of light is *one lumen for one second.*
Name: ......
Symbol: ......

VI. *Diffusion constant*, is defined as *the ratio of the brightness to the illumination at any point on a surface.*

The numerical value of this constant represents the fraction of the incident light, at any point, which is diffusely reflected by the surface, through unit solid angle.

Sometimes brightness is defined differently from the manner in which it has been defined above, viz: to denote the intensity (instead of the luminous current) of unit area. In this case, the *diffusion constant* becomes the ratio between 2π *brightness* and the *illumination*. That is, if we denote the diffusion constant by "N," its defining equation is

$$N=2\frac{\pi \text{ brightness}}{\text{illumination}}$$

In the system of units which we have employed above, however,

$$N=\frac{\text{brightness}}{\text{illumination}}$$

VII.

LUMINOSITY.

In all that has been said above, it has been tacitly assumed that the quantities under consideration, (*brightness*, *illumination*, *etc.*) refer to luminous energy of the same quality, *i. e.*, to lights of the same composition, or colors of the same hue. But, in practice, it becomes very frequently necessary to compare lights of different color.

In general, indeed, the brightness of the wall of a room has a different hue from that of the illumination which produces this brightness.

Accordingly it becomes necessary for us to define just what we mean when we say that a certain room illuminated by *blue* light is just as brilliantly lighted as a certain other room which is illuminated by *red* light.

The ease with which one can read a newspaper depends not only upon the intensity of the light with which it is illuminated, but very largely upon the quality of this light. That particular property of any color which determines its value *as an illuminant* is called its "*luminosity*."

Thus, for the normal eye, yellow light is much more useful than red of the same intensity; and red light, in turn, is a more powerful aid to distinct vision than blue of the same intensity.

It only remains now to give to "*luminosity*" a quantitive definition. This is done by the use of a principle discovered by Rood [*Amer. Jour. Sci.* Vol. 46, 3rd Series, p. 173. (1893)], viz: that when a normal eye is allowed to perceive a colored surface for a short interval of time, say a fraction of a second, the intensity of the sensation is independent of the hue and *depends only upon the luminosity*. If a circular cardboard disc be covered, one half with gray, the other half with a colored pigment, the two halves will have equal luminosities when on rotation all sense of flickering disappearing. It is thus found that each color in the spectrum requires a definite gray to "match" it. The amount of white in the gray semi-circle which matches any given color is a measure of the luminosity of this color. In this connection it may be added that any color is completely defined only when we know the following three things about it, viz:

1. Its "hue," *i. e.*, the wave length of the light in the solar spectrum which most nearly matches it.
2. Its "*luminosity*."
3. Its dilution, or "*purity*," *i. e.*, the amount of white light which is mixed with the pure color producing it.